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Generalized Pure Modules

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For right modules $M < N$ over a ring R, consider any system of equations in M of the form $\sum\{x_i r_{ij} \mid i \in I\} = d_j \in M$, $j \in J$, where $r_{ij} \in R$. The usual definition of M as *pure* in N is that for any such a finite system, if the system is solvable in the bigger module N, then it is already solvable in M. Here the above ordinary concept of purity will be generalized by allowing I and J to be of possibly infinite cardinalities $|I| < \mu$ and $|J| < \aleph$ for fixed cardinals μ and $\aleph$. In this way, generalized $(\mu^<, \aleph^<)$-pure and absolutely pure concepts are defined in terms of μ and $\aleph$ and studied. Here the number $\aleph$ of relations of a module is simultaneously studied with the more familiar number μ of generators. © 2001 Academic Press

INTRODUCTION

Every module M can be represented in terms of generators and relations $M = \langle y_i, i \in I \mid \sum_{i\in I} y_i r_{ij} = 0, j \in J\rangle$; M is $(\mu^<, \aleph^<)$-presented if $|I| < \mu$ and $|J| < \aleph$, where μ and $\aleph$ are finite, or more importantly infinite cardinals. The first Theorem 2.10 gives some condition on the relations matrix $\|r_{ij}\|$ which induces a direct sum decomposition of the module M. Noteworthy is the case $\mu = 2$, when we have only a single unknown and lots of equations. Usually the so-called finite case $(\aleph_0^<, \aleph_0^<)$ refers to well-known classical concepts and theorems.

A module A is classically absolutely pure, if whenever $A \hookrightarrow B$ embeds as a submodule in a bigger module B, then A is necessarily pure in B. Section 3 extends the known theory of $(\aleph_0^<, \aleph_0^<)$-absolutely pure (i.e., absolutely pure) modules to absolutely $(\mu^<, \aleph^<)$-pure modules. For example, it was first shown for commutative Dedekind rings in [Ma], and then later

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in general in [ES] and [Me], that the ring R is Noetherian if and only if every absolutely pure module is injective. Here Theorem 3.9 generalizes this the rest of the way for all infinite cardinals $\aleph \geq \aleph_0$ by showing that R is $\aleph^<$-Noetherian if and only if every absolutely $(\mu^<, \aleph^<)$-pure module is injective, where μ is arbitrary $2 \leq \mu \leq \aleph$. Theorem 3.5 shows that A is absolutely $(\mu^<, \aleph^<)$-pure if and only if any consistent system of strictly less than $\aleph$ equations with strictly less than μ unknowns over A is solvable in A. So far such absolutely $(\mu^<, \aleph^<)$-pure modules for both $\aleph < \aleph_0$ and $\mu < \aleph_0$ finite have not been investigated much. However, many of the results in Section 3 apply to them just as well.

Theorem 3.5 also shows that a module A is absolutely $(\mu^<, \aleph^<)$-pure if and only if $\mathrm{Ext}_R^1(M, A) = 0$ for every $(\mu^<, \aleph^<)$-presented module M. The more difficult converse, in which A is allowed to vary, while M is fixed is proved in Theorem 3.12. This converse for the finite $(\aleph_0^<, \aleph_0^<)$-case had been investigated before in [E]. Examples of absolutely $(\mu^<, \aleph^<)$-pure modules are provided in 3.11, and more importantly in Construction 3.8 (4). The latter already indicates the key role played by the $\aleph^<$-ascending chain condition on R, which is further explored in Section 4. Theorem 3.15 gives a condition on cardinalities of the numbers of generators and relations of certain submodules of free modules which guarantees that every reduced product (see Definition 1.4) of injective modules modulo appropriate filters is absolutely $(\mu^<, \aleph^<)$-pure. Theorem 3.15 is technically challenging to prove. In a special case with $\mu = 2$ and $\aleph = \aleph_0$ it was shown by Eklof and Sabbagh (see [ES] and 3.16) that it forces the ring R to be coherent.

Section 4 and Theorem 4.2 gives us a second new characterization of $\aleph^<$-Neotherian rings for regular $\aleph_0 \leq \aleph$. A ring R is $\aleph^<$-Noetherian if and only if for any module M, if the number of generators $\mathrm{gen}\, M = |I|$ is less than $\aleph$, then M is also $\aleph^<$-related; i.e., $\mathrm{rel}\, M = |J| < \aleph$ (see Definition 2.6).

Throughout here $\aleph^<$-products and reduced products are used (1.4). Speaking loosely and imprecisely, in the finite case (of Goldie dimensions, Noetherian, purity, etc.) direct sums are used, while in higher cardinal analogues, or sometimes generalizations, at least $\aleph^<$-products are needed, as is amply evidenced in [DF1 and DF2], [Lo1–Lo4], [LLS], [T], and [D1–D3]. The $\aleph^<$-ascending chain condition for infinite cardinals has been studied in [KM], [Lo2–Lo4], [T], and [D2 and D3]. Although here the considerations were limited to the $\aleph^<$-ascending chain condition, other generalized finiteness-type hypotheses on R or on modules have been considered in [EM] and [Lo1–Lo4]. It is beyond the scope of this article to systematically formulate a theory of pure $(\mu^<, \aleph^<)$-injective modules, $(\mu^<, \aleph^<)$-compact modules, $(\mu^<, \aleph^<)$-pure short exact sequences, and $(\mu^<, \aleph^<)$-semihereditary and coherent modules. The author hopes to return to these topics in the near future.

1. PRELIMINARIES

Reduced products modulo a filter $\mathcal{F}$ on the index set of a product, and the completeness cpl$\mathcal{F}$ of $\mathcal{F}$ are defined.

1.1. *Notation*. Modules are right unital over an associative ring R. Submodules are denoted by "$<$," "$\leq$," "$\subset$," or "$\subseteq$;" large or essential ones by "$\ll$;" "$A <\not\leq B$" means that $A < B$ is a nonessential extension. For $m \in M$ and $K < M$, $m^{\perp} = \{r \in R \mid mr = 0\}$, $m^{-1}K = \{r \mid mr \in K\} = (m+K)^{\perp} \leq R$, $K^{\perp} = \{r \mid Kr = 0\} \triangleleft R$, where "$\triangleleft$" denotes ideals. The notation $A \hookrightarrow B$ means that there exists an embedding of A into B.

The injective hull over R of M is written both as $\widehat{M}$, or as $E(M) = EM$ when M is given by a complicated formula. The submodule generated by a subset $Y \subset M$ is denoted by $\langle Y \rangle = \sum\{yR \mid y \in Y\}$.

1.2. *Notation* (Cardinals). For any set I, $|I|$ will denote its cardinality, and thus for the set $\mathcal{P}(I)$ of all subsets of I, $|\mathcal{P}(I)| = 2^{|I|}$. For singular and regular cardinals, see [HJ, p. 193, Def. 2.6]. For any limit ordinal $\aleph$, $\operatorname{cof}\aleph = \operatorname{cof}(\aleph)$ is its cofinality ([HJ, pp. 195–196, Def. 2.6]). Define $\aleph^{+} = |\aleph|^{+}$ to be the successor cardinal of any cardinal $\aleph = |\aleph|$.

1.3. *Notation* (Filters). For any infinite set I, and a filter $\mathcal{F} \subseteq \mathcal{P}(I)$ on I ([Lo 3, p. 74] or [HJ, p. 202, Def. 1.1]), for a cardinal $\aleph$, $\mathcal{F}$ is $\aleph^{<}$-complete if for any subset $J \subseteq \mathcal{F}$, $|J| < \aleph$, $\cap J \in \mathcal{F}$; otherwise $\mathcal{F}$ is $\aleph^{<}$-incomplete. Define $\operatorname{cpl}(\mathcal{F})$, to be the unique smallest cardinal such that $\mathcal{F}$ is $\operatorname{cpl}(\mathcal{F})^{+<}$-incomplete. That is, $\operatorname{cpl}(\mathcal{F})$ is the smallest cardinal such that there exists a $J \subseteq \mathcal{F}$, $|J| = \operatorname{cpl}(\mathcal{F})$, but $\cap J \notin \mathcal{F}$. An equivalent definition is that $\operatorname{cpl}(\mathcal{F})$ is the largest cardinal such that $\mathcal{F}$ is $\operatorname{cpl}(\mathcal{F})^{<}$-complete. Always $\operatorname{cpl}(\mathcal{F}) \geq \aleph_0$.

1.4. *Notation* (Reduced Products). For an ordinal $\aleph_0 \leq \aleph$ and modules M_i, $i \in I$, their $\aleph^{<}$-*product* is $\prod^{<\aleph}\{M_i \mid i \in I\} = \prod_{i\in I}^{<\aleph} M_i = \prod^{<\aleph} M_i = \{x = (x_i)_{i\in I} \in \prod_{i\in I} M_i \mid |\operatorname{supp} x| < \aleph\}$ where the *support* of x is $\operatorname{supp} x = \{i \in I \mid x_i \neq 0\}$. We use the convention that for $\aleph = \infty$, an $\aleph^{<}$-product is the whole product.

For a given filter $\mathcal{F} \subseteq \mathcal{P}(I)$ on I, for $x = (x_i)_{i\in I}$, $y = (y_i)_{i\in I} \in \prod_{i\in I} M_i$, define $x \sim y$ if $\{i \mid x_i = y_i\} \in \mathcal{F}$. Then "$\sim$" is a congruence relation, and the *reduced product* $\prod_{i\in I} M_i/\mathcal{F} = \prod_{i\in I} M_i/\sim$ is defined as the product modulo this congruence relation. If $\mathcal{F}$ is an ultrafilter the reduced product is called an *ultraproduct*. Define $\prod_{i\in I}^{\mathcal{F}} M_i = \{x \in \prod_{i\in I} M_i \mid x \sim 0\} = \{x = (x_i)_{i\in I} \mid I\backslash \operatorname{supp} x \in \mathcal{F}\}$. Then there is a short exact sequence of modules ([Lo 3, p. 74])

$$0 \longrightarrow \prod_{i\in I}^{\mathcal{F}} M_i \longrightarrow \prod_{i\in I} M_i \longrightarrow \prod_{i\in I} M_i/\mathcal{F} \longrightarrow 0.$$

Note that $\mathscr{G} = \{J | J \subseteq I, |I \backslash J| < \aleph\}$ is a filter, and $\prod_{i\in I}^{\mathscr{G}} M_i = \prod_{i\in I}^{<\aleph} M_i$. Also, if $\aleph_0 \leq \aleph \leq |I|$ and $\aleph$ is regular, then $\operatorname{cpl}(\mathscr{G}) = \aleph$.

2. SYSTEMS OF EQUATIONS

Systems of equations over a module are defined, properties of submodules and modules in terms of degrees of solvability of these equations, and subsystems are defined. Theorem 2.10 here shows how the equations used in the presentation of a module M determine or induce a direct sum decomposition of the module M.

2.1. *Notation*. Let I and J be index sets of arbitrary cardinalities $|I|, |J|$ except that $|I| \leq \aleph_0 |J|$ and if J is finite, so is I. Consider

$$\mathscr{S} : \sum_{i\in I} x_i r_{ij} = d_j \in M, \ j \in J, \ r_{ij} \in R, \ \|r_{ij}\| \text{ is column finite.}$$

Then $\mathscr{S}$ will be referred to as a *system* of equations over the module M, or just a system, or just $\mathscr{S}$.

The system $\mathscr{S}$ is *consistent* if the following holds. Let $F \subset J$ be a finite set and $\{c_j \mid j \in F\} \subset R$ such that $\sum_{j\in F} r_{ij} c_j = 0$ for all i, or all i in the finite set $\{i \mid \exists\, j, c_j \neq 0 \text{ and } r_{ij} \neq 0\}$. Then necessarily also $\sum_{j\in F} d_j c_j = 0$. The concept that two systems $\mathscr{S}, \mathscr{P}$ over M are *equivalent* is defined by a similar extension of the definition from the finite case.

2.2. Definition. Let $2 \leq \mu, \aleph$ be cardinals, or ∞, where "∞" is a (noncardinal) symbol larger than any cardinal, with $\mu \leq \aleph_0 \cdot \aleph$ (where $\aleph_0 \cdot \infty = \infty$), and if $\aleph < \aleph_0$, then also $\mu < \aleph_0$. A submodule $M < N$ is $(\mu^<, \aleph^<)$-*pure* if for any, or all, systems $\mathscr{S}$ with $|I| < \mu$ and $|J| < \aleph$, whenever $\mathscr{S}$ is solvable in N, then it also is solvable in M.

The module A is *absolutely* $(\mu^<, \aleph^<)$-*pure*, also sometimes called $(\mu^<, \aleph^<)$-*absolutely pure*, if any extension of modules $A < B$ is $(\mu^<, \aleph^<)$-*pure* for all B and $A \hookrightarrow B$.

Relative to any kind of concept of pure submodule whatever, there is always automatically a related corresponding notion as in the next definition.

2.3. Definition. A short exact sequence of modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is $(\mu^<, \aleph^<)$-*pure* if the image of A in B is.

2.4. *Remark*. With both $\mu = \aleph$ equal, a few of the latter types of definitions have been made and studied by several authors. Only some of these are [EM, ES, JL, and La].

2.5. *Remark* (Special Cases). (1) In the *finite case* of $(\aleph_0^<, \aleph_0^<)$, we have a finite number of equations $\mathcal{S}$. Then the above definitions reduce to the usual old accepted definitions of pure submodule, pure extension of modules, absolutely pure module, and pure short exact sequence.

(2) If $\aleph = \mu = \infty$, there are no cardinality restrictions on $\mathcal{S}$, although once an $\mathcal{S}$ is given, then I, J, and $\mathcal{S}$ are sets with some definite cardinality. A module M is $(\infty^<, \infty^<)$-absolutely pure if and only if M is injective.

(3) To avoid trivialities we always assume $2 \leq \mu$, $2 \leq \aleph$ throughout.

The completeness requirement below is an integral part of the definition of a presentation.

2.6. Definition. For a module M, a *presentation* p of M is a triple $\langle I \times J, \{y_i\}_{i\in I}, \|r_{ij}\|\rangle$ with $M = \sum_{i\in I} y_i R = \langle y_i, i \in I \mid \sum_{i\in I} y_i r_{ij} = 0, j \in J\rangle$, where the relations given by the matrix $\|r_{ij}\|$ satisfy the following completeness requirement. If $\sum_{i\in I} y_i a_i = 0$ for a finite number of nonzero $a_i \in R$, then there exists a finite number of nonzero $t_j \in R$ with $a_i = \sum_{j\in J} r_{ij} t_j$, for all $i \in I$.

As usual, $\operatorname{gen} M$ denotes the minimal cardinality of a generating set of M. For any presentation p of M, define $\operatorname{gen} p = |I|$ and $\operatorname{rel} p = |J|$. Define $\operatorname{rel} M = \text{minimum}\{\operatorname{rel} p \mid p \text{ is a presentation of } M\}$. For cardinals $\mu, \aleph, M$ is $(\mu^<, \aleph^<)$-*presented* if M has a presentation p with $|I| < \mu$, $|J| < \aleph$; and M is *finitely presented* if it is $(\aleph_0^<, \aleph_0^<)$-presented.

If for a module N, $\operatorname{gen} N > \operatorname{rel} N$, then $N = M \oplus F$, where F is free, and $\operatorname{gen} M \leq \aleph_0 \cdot \operatorname{rel} M$. This means either that both $\operatorname{rel} M$ and $\operatorname{gen} M$ are finite or that $\aleph_0 \leq \operatorname{rel} M$, in which case $\operatorname{gen} M \leq \operatorname{rel} M$. Hence here we frequently assume that $\mu \leq \aleph_0 \cdot \aleph$, and if $\aleph < \aleph_0$ then also $\mu < \aleph_0$.

2.7. Definition. For a cardinal $\aleph \geq \aleph_0$, a module M is $\aleph^<$-*Noetherian*, or satisfies the $\aleph^<$-*ascending chain condition* (abbreviation: $\aleph^<$-A.C.C.), if any ordinal indexed strictly ascending chain of submodules $M_0 \subset M_1 \subset \cdots \subset M_\alpha \subset \cdots \leq M$, $\alpha < \tau$ has strictly less than $\aleph$-terms; i.e., $\tau < \aleph$. Let $\sigma(R)$ be the unique smallest cardinal such that R as a right R-module satisfies the $\sigma(R)^<$-A.C.C.

2.8. *Remarks.* (1) Note that $\operatorname{rel} M \leq \text{minimum}\{\operatorname{rel} p \mid p \text{ is a presentation of } M \text{ with } \operatorname{gen} p = \operatorname{gen} M\}$. (2) The author does not have an example showing that the next lemma is false for finitely generated modules.

2.9. Lemma. *If for a module M, $\operatorname{gen} M \geq \aleph_0$, then there exists a presentation p of M as above in* 2.6 *with* $\operatorname{gen} p = \operatorname{gen} M = |I|$, *and* $\operatorname{rel} p = \operatorname{rel} M = |J|$.

Proof. Select any presentation of $M = \langle y_i, i \in I \mid \sum_{i\in I} y_i r_{ij} = 0,\ j \in J\rangle$ with $\operatorname{rel} M = |J|$ minimal. Let $M = \langle \{x_p \mid p \in P\}\rangle$ be any minimal cardinality generating set of M. For each $p \in P$, write $x_p = \sum_{i\in I} y_i s_{ip}$, $s_{ip} \in R$. Define $I(0) = \{i \in I \mid \exists\ p \in P, s_{ip} \neq 0\}$. Then $|I(0)| = |P| = \operatorname{gen} M$. For each $k \in I\backslash I(0)$, we have $y_k = \sum_{i\in I(0)} y_i g_{ik}$ for some $g_{ik} \in R$. For any $j \in J$,

$$\sum_{i\in I} y_i r_{ij} = \sum_{i\in I(0)} y_i r_{ij} + \sum_{i\in I(0)} y_i \left(\sum_{k\in I\backslash I(0)} g_{ik} r_{kj} \right) = \sum_{i\in I(0)} y_i f_{ij},$$
$$f_{ij} = r_{ij} + \sum_{k\in I\backslash I(0)} g_{ik} r_{kj}.$$

In order to show that the latter expressions generate all the relations on the minimal cardinality generating set $\{y_i \mid i \in I(0)\}$, suppose that $\sum_{i\in I(0)} y_i a_i = \sum_{i\in I} y_i a_i = 0$ for $a_i \in R$, with $a_k = 0$ for $k \in I\backslash I(0)$. Then for some finite number of $b_j \in R$

$$a_i = \sum_{j\in J} r_{ij} b_j,\ i \in I(0);\ \ a_k = 0 = \sum_{j\in J} r_{kj} b_j,\ \ k \in I\backslash i(0).$$

Consequently, for $i \in I(0)$,

$$a_i = \sum_{j\in J} r_{ij} \left(b_j + \sum_{k\in I\backslash I(0)} g_{ik} r_{kj} b_j \right) = \sum_{j\in J} f_{ij} b_j.$$

Thus, $M = \langle y_i, i \in I(0) \mid \sum_{i\in I(0)} y_i f_{ij} = 0, j \in J\rangle$ is a presentation with $\operatorname{gen} M = |I(0)|$ and $\operatorname{rel} M = |J|$. ∎

The next theorem illustrates the theme of this article that not only the generators of a module are important, but also the relations count. In it there are no irreduncancy assumptions made about the nonzero columns of the relations matrix. In a presentation of a module M, a zero i-th row in the relations matrix $\|r_{ij}\|$ implies that $R \cong y_i R$ is a free direct summand of M.

2.10. THEOREM. *For any module M and any presentation*

$$M = \left\langle y_i, i \in I \mid \sum_{i\in I} y_i r_{ij} = 0, j \in J \right\rangle \textit{ with all } y_i \neq 0, \textit{ define}$$
$$\nu = \sup_{i\in I} |\{j \mid r_{ij} \neq 0\}|.$$

Assume that there are no zero columns or rows in $\|r_{ij}\|$. Then for some ordinal

λ there exist partitions $I = \cup\{I_k \mid k < \lambda\}$, $J = \cup\{J_k \mid k < \lambda\}$ and well orderings of I and J such that the

(i) *$I \times J$ matrix $\|r_{ij}\|$ is block rectangular with rectangular $I_k \times J_k$ submatrices along the main diagonal with zeros everywhere else, where for any $k < \lambda$.*

$$|I_k| \leq \nu \cdot \aleph_0, \quad |J_k| \leq \nu \cdot \aleph_0;$$

(ii) *$\forall\, i \in I_k$, $\{j \mid r_{ij} \neq 0\} \subseteq J_k$, and $\forall\, j \in J_k$, $\{i \mid r_{ij} \neq 0\} \subseteq I_k$, $k < \lambda$.*

(iii) *$M = \bigoplus_{k<\lambda} M_k$, where $M_k = \langle y_i, i \in I(k) | \sum\{y_i r_{ij} \mid i \in I_k\} = 0, j \in J_k\rangle$.*

(iv) *$|\lambda| \leq \min(|I|, |J|)$; $|I|, |J| \leq |\lambda| \cdot \nu \cdot \aleph_0$.*

Proof. Define $f : \mathcal{P}(I) \longrightarrow \mathcal{P}(J)$, and $g : \mathcal{P}(J) \longrightarrow \mathcal{P}(I)$ for all $A \subseteq I$ and $B \subseteq J$ by $f(A) = \{j \in J \mid \exists\, a \in A, r_{aj} \neq 0\}$ and $g(B) = \{i \in I \mid \exists\, b \in B, r_{ib} \neq 0\}$. If $A \neq \emptyset$ and $B \neq \emptyset$, then also $f(A) \neq \emptyset$ and $g(B) \neq \emptyset$. Note that $A \subseteq (gf)^n(A)$ and $B \subseteq (fg)^n(B)$ for any $n = 0, 1, 2, \ldots$. Well order I and let $i(0) \in I$ be the smallest element. Define $I_0 = \cup_{n<\omega}(gf)^n(\{i(0)\})$ and $J_0 = \cup_{n<\omega}(fg)^n[f(\{i(0)\})]$. Then $J_0 = f(I_0) = fg(J_0)$, $I_0 = g(J_0) = gf(I_0)$, and I_0, J_0 satisfy (ii). First, $|g(B)| \leq |B| \cdot \aleph_0$ while $|f(A)| \leq |A| \cdot \nu$. Thus $|(fg)^n(B)| \leq |B| \cdot \nu \cdot \aleph_0$ and $|(gf)^n(A)| \leq |A| \cdot \nu \cdot \aleph_0$ for all $n = 0, 1, 2, \ldots$. Hence $|I_0| \leq \nu \cdot \aleph_0$ and $|J_0| \leq \nu \cdot \aleph_0$ satisfy (i). Assume that for some ordinal β, for all $\alpha < \beta$, $i(\alpha) \in I_\alpha$, I_α, and J_α have already been selected as above for $\alpha = 0$. Let $i(\beta) \in I \setminus \cup_{\alpha<\beta} I_\alpha$ be the smallest element, and then construct I_β and J_β exactly as above except with $\{i(0)\}$ now replaced by $\{i(\beta)\}$. Note that $I_\beta \cap I_\alpha = \emptyset$, as well as $J_\beta \cap J_\alpha = \emptyset$ for all $\alpha < \beta$. By ordinal introduction, there exists a smallest ordinal λ such that $I = \cup_{k<\lambda} I_k$ and $J = \cup_{k<\lambda} J_k$ satisfy (i) and (ii). Well order both I and J so that $||r_{ij}||$ becomes block diagonal.

(iii) Define $G = \langle \sum_{i\in I} x_i r_{ij} \mid j \in J\rangle < F = \oplus_{i\in I} x_i R$, and identically $G_k < F_k$ for the index sets I_k, J_k for $k < \lambda$. Then $G = \oplus_{k<\lambda} G_k < \oplus_{k<\lambda} F_k = F$. There are two canonical isomorphisms $\oplus_{k<\lambda} F_k/G_k \longrightarrow F/G \longrightarrow M$ defined by $x_i + G_k \longrightarrow x_i + G \longrightarrow y_i$ whose composite maps $F_k/G_k \cong M_k$. Consequently, $M = \oplus_{k<\lambda} M_k$.

(iv) Since $J = \cup_{k<\lambda} J_k$ is a partition into nonempty subsets J_k, $|\lambda| = \sum_{k<\lambda} 1 \leq \sum_{k<\lambda} |J_k| = |J|$. Similarly, $|\lambda| \leq |I|$, and hence $|\lambda| \leq \min(|I|, |J|)$. Lastly from (i), $|J| = \sum_{k<\lambda} |J_k| \leq \sum_{k<\lambda} \nu \cdot \aleph_0 = |\lambda| \cdot \nu \cdot \aleph_0$, and likewise $|I| \leq |\lambda| \cdot \nu \cdot \aleph_0$. ∎

Just by looking at the two cardinals $|J|$ and ν defined by the relations matrix $\|r_{ij}\|$ of a module M sometimes we can tell that the module M is highly decomposable.

2.11. COROLLARY. *Assume that M in the last theorem satisfies $\aleph_0 < |J|$ and $\nu < |J|$. Then*

(i) $|I| = |J| = |\lambda|$.

(ii) $M = \oplus_{k<|\lambda|} M_k, 0 \neq M_k < M$, *all* k.

Proof. By 2.10 (iv), $|\lambda| \leq |J| \leq |\lambda| \cdot \nu \cdot \aleph_0$. If $|\lambda| < |J|$, then since $\nu \cdot \aleph_0 < |J|$, also $|\lambda| \cdot \nu \cdot \aleph_0 < |J|$, a contradiction. Hence $|J| = |\lambda|$. Also by 2.10 (iv), $|\lambda| \leq |I| \leq |\lambda| \cdot \nu \cdot \aleph_0$. If $|\lambda| < |I|$, then $\nu \cdot \aleph_0 < |J| = |\lambda| < |I|$, and consequently $|\lambda| \cdot \nu \cdot \aleph_0 < |I|$. Thus also $|I| = |\lambda|$.

3. PURITY

Some of the properties of $(\mu^<, \aleph^<)$-pure submodules $A < M$ are derived. Absolutely $(\mu^<, \aleph^<)$-pure modules are related to $(\mu^<, \aleph^<)$-presented modules; properties of such modules are related to properties of the ring.

The next proposition is a partial generalization of the well-known abelian group theorem that $A < M$ is pure if and only if for any abelian subgroup $A \subset B \subseteq M$ such that B/A is finitely generated, A is a summand of B ([F, Vol. 1, p. 121, Theorem 28.4, (i) $\Longleftrightarrow$ (iii)]. It should be noted that the latter, as well as [B, pp. 357–358], and [G, pp. 89–91] somewhere in their proofs use the abelian group theorem that $A < M$ is pure if and only if every coset modulo A contains an element of the same order as this coset. However, in the special case $\mu = \aleph = \infty$, the proof given in [G, Theorem 2, pp. 90–91] for abelian groups could be generalized to prove the next proposition. It seems to be both new and curious that it also holds for $\mu < \aleph_0$ and $\aleph < \aleph_0$.

3.1. PROPOSITION. *For extended cardinals $\mu \leq \infty$, $\aleph \leq \infty$, $\mu \leq \aleph_0 \cdot \aleph$, and if $\aleph < \aleph_0$, then $\mu < \aleph_0$, $A < M$ is $(\mu^<, \aleph^<)$-pure $\Rightarrow \forall\, B, A \subset B \leq M$, if B/A has a $(\mu^<, \aleph^<)$-presentation, then $B = A \oplus C$ for some $C \leq B$.*

Proof. Write $B/A = \langle b_i + A, i \in I \mid \sum_{i \in I} b_i r_{ij} = d_j \in A, j \in J \rangle$ with $|I| < \mu$, $|J| < \aleph$. The system $\sum_{i \in I} x_i r_{ij} = d_j$, $j \in J$ has a solution $x_i = a_i \in A$. Set $c_i = b_i - a_i$ and $C = \langle c_i, i \in I \rangle = \sum_{i \in I} c_i R \leq B$. Map $\pi : B \longrightarrow B/A$, and define $\rho : B/A \longrightarrow B$ by $\rho(\sum_{i \in I} b_i r_i + A) = \sum_{i \in I} c_i r_i$. If $\sum_{i \in I} b_i r_i \in A$, then there exists a finite number $t_j \in R$ such that $r_i = \sum_{j \in J} r_{ij} t_j$, and hence also $\sum_{i \in I} c_i r_i = 0$. Thus ρ is well defined with $\rho\pi = 1$, and $A = \ker \pi < B$ is a direct summand. ∎

The next three lemmas will be needed to prove the next theorem. Note that the first lemma can also be formulated in terms of the projective property relative to a short exact sequence (see [D4, Theorems 18–27, pp. 371, 373–374]).

3.2. LEMMA. *For $\mu, \aleph \leq \infty$ as in 2.2, and modules $A < B$, let $\pi : B \longrightarrow B/A$ be the natural projection. Then $A < B$ is $(\mu^<, \aleph^<)$-pure if and only if for every $(\mu^<, \aleph^<)$-presented module M, the map $\pi^* : \mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, B/A)$ is onto.*

3.3. LEMMA. *For any modules M, A let $\pi : \widehat{A} \longrightarrow \widehat{A}/A$ be the natural projection. Then $\pi^* : \mathrm{Hom}_R(M, \widehat{A}) \longrightarrow \mathrm{Hom}_R(M, \widehat{A}/A)$ is onto if and only if $\mathrm{Ext}^1_R(M, A) = 0$.*

3.4. LEMMA. *For any free module F and submodule $G < F$, let $\alpha : G \longrightarrow F$ be the inclusion and let A be any fixed given module. Then the induced map $\alpha^* : \mathrm{Hom}(F, A) \longrightarrow \mathrm{Hom}_R(G, A)$ is onto if and only if $\mathrm{Ext}^1_R(F/G, A) = 0$.*

Various previous facts about absolutely pure modules can now be assembled in a comprehensive generalized form into a theorem characterizing absolute $(\mu^<, \aleph^<)$-purity. Ordinary absolute purity is the case $\mu = \aleph = \aleph_0$, while $(\infty^<, \infty^<)$-absolutely pure modules are injective. Note that below in 3.5(4), if for any $G \leq F$, $\operatorname{gen} G < \aleph$, then the module A is F-injective ([MM, p. 1, Def. 1.1]).

3.5. THEOREM. *Let $\mu \leq \infty$, $\aleph \leq \infty$ with $\mu \leq \aleph_0 \cdot \aleph$ be extended cardinals with $\mu < \aleph_0$ in case $\aleph < \aleph_0$, and let A be a module over a ring R. The following are all equivalent.*

(1) *$A < \widehat{A}$ is $(\mu^<, \aleph^<)$-pure.*

(2) *A is absolutely $(\mu^<, \aleph^<)$-pure.*

(3) *$\forall$ $(\mu^<, \aleph^<)$-presented module M, $\mathrm{Ext}^1_R(M, A) = 0$.*

(4) *$\forall$ $(\mu^<, \aleph^<)$-presented module F/G, where F is free, $\operatorname{gen} F < \mu$, $\operatorname{gen} G < \aleph$, every R-map $\beta : G \longrightarrow A$ extends to $\gamma : F \longrightarrow A$ with $\gamma|G = \beta$.*

(4′) *$\forall$ projective P with $\operatorname{gen} P < \mu$ and $\forall$ $G < P$ with $\operatorname{gen} G < \aleph$, the map extension property of (4) holds.*

(5) *Any consistent $(\mu^<, \aleph^<)$-system of equations over A has a solution in A.*

Proof. (1) $\Longrightarrow$ (2). We have to show that any extension $A < B$ is $(\mu^<, \aleph^<)$-pure. From $A < \widehat{A} \leq \widehat{B} = \widehat{A} \oplus C$ for some $C < \widehat{B}$. Since $A < \widehat{A}$ and $\widehat{A} < \widehat{B}$ are $(\mu^<, \aleph^<)$-pure, by transitivity, so is $A < \widehat{B}$. But for any $A < D \leq \widehat{B}$, and in particular for $D = B$, $A < B$ is $(\mu^<, \aleph^<)$-pure. (2) $\Longrightarrow$ (1) is trivial.

Next, to show that (2) $\Longleftrightarrow$ (3), replace it by (1) $\Longleftrightarrow$ (3), and then use 3.2 and 3.3 to conclude that

A is absolutely $(\mu^<, \aleph^<)$-pure $\Longleftrightarrow$

$\Longleftrightarrow$ $\pi^* : \mathrm{Hom}_R(M, \widehat{A}) \longrightarrow \mathrm{Hom}_R(M, \widehat{A}/A)$ is onto $\Longleftrightarrow$ $\mathrm{Ext}^1_R(M, A) = 0$.

(4) $\Longrightarrow$ (4′). For a free module F, $F = P \oplus C$ with $\operatorname{gen} F = \operatorname{gen} P$. The rest is clear.

(4) $\Longleftrightarrow$ (5). For any system $\mathcal{S} : \sum_{i\in I} x_i r_{ij} = d_j \in A$, $j \in J$, form $G = \langle \sum_{i\in I} x_i r_{ij} \mid j \in J\rangle < F = \oplus_{i\in I} x_i R$, where F is free on $\{x_i\}_{i\in I}$ and set $\beta(\sum_{i\in I} x_i r_{ij}) = d_j$. Then $\mathcal{S}$ is consistent if and only if β extends to an R-module homomorphism $\beta : G \longrightarrow A$, which then is obtained by extending β to be R-linear on all of G. For a choice $\{a_i\}_{i\in I} \subset A$, the assignment $\gamma x_i = a_i$, $i \in I$ defines an R-map $\gamma : F \longrightarrow A$. But then $x_i = a_i$, $i \in I$ is a solution of the above $\mathcal{S}$ if and only if $\gamma|G = \beta$. The latter shows that (4) $\Longleftrightarrow$ (5), or also that (2) $\Longleftrightarrow$ (5).

(3) $\Longleftrightarrow$ (4). In (4), let $\alpha : G \hookrightarrow F$ and $\alpha_* : \operatorname{Hom}_R(F, A) \longrightarrow \operatorname{Hom}_R(G, A)$. By 3.5 the map α_* is onto if and only is $\operatorname{Ext}^1_R(F/G, A) = 0$ if and only if for any β there exists a γ with $a_*(\gamma) = \gamma\alpha = \beta$. ∎

The result [ES, p. 258, Lemma 3.4] follows by taking $\mu = 2$ in 4.2(4) above, in which case $G = L < R = F$. It should be stressed that their latter result as formulated in [ES, p. 257–258, Lemmas 3.2 and 3.4] applies equally well to the finite $2 \le \aleph < \aleph_0$ case as well as to the infinite case $\aleph_0 \le \aleph$, and that so also does the last theorem. Note that below (ii) for $\aleph = \infty$ is Baer's criterion, and consequently the absolutely $(2^<, \infty^<)$-pure modules in (i) are exactly the injectives.

3.6. COROLLARY 1. (P. Eklof and G. Sabbagh). *For $\aleph \le \infty$, the following are equivalent for an R-module A*:

(i) *A is absolutely $(2^<, \aleph^<)$-pure.*

(ii) *For any $L < R$ with $\operatorname{gen} L < \aleph$ any homomorphism $\varphi : L \longrightarrow A$ extends to $\bar{\varphi} : R \longrightarrow A$ with $\bar{\varphi}|L = \varphi$.*

Use of 2.6 *and* 3.5(3) *gives an alternate characterization of the above modules.*

3.7. COROLLARY 2. *For $\aleph \le \infty$, for a module A*, (1) $\Longleftrightarrow$ (2), *where*

(1) *A is absolutely $(2^<, \aleph^<)$-pure.*

(2) *For any $L < R$ if $\operatorname{gen} L < \aleph$, then $\operatorname{Ext}^1_R(R/L, A) = 0$.*

Finally, below (4) *is an encouraging result; it gives a very explicit method of construction absolutely $(\mu^<, \aleph^<)$-pure modules which are not injective over any non-Noetherian ring.*

3.8. DEFINITION (Construction). For $\aleph_0 \le \aleph < \infty$, $\mu \le \aleph$, for any family of $(\mu^<, \aleph^<)$-pure submodules $A_\gamma < M_\gamma$, $\gamma \in \Gamma$, let $\mathcal{F}$ be any filter in $\mathcal{F} \subseteq \mathcal{P}(\Gamma)$ with $\operatorname{cpl}(\mathcal{F}) \ge \aleph$. Set $A = \prod^{\mathcal{F}}_{\gamma\in\Gamma} A_\gamma < M = \prod_{\gamma\in\Gamma} M_\gamma$. In particular, if $\aleph$ is regular we may let $A = \prod^{<\aleph}_{\gamma\in\Gamma} A_\gamma$. Then

(1) $A < M$ is $(\mu^<, \aleph^<)$-pure.

(2) If A_γ, $\gamma \in \Gamma$ are absolutely $(\mu^<, \aleph^<)$-pure, then A is absolutely $(\mu^<, \aleph^<)$-pure.

(3) In particular, if all A_γ, $\gamma \in \Gamma$ are injective, then A is absolutely $(\mu^<, \aleph^<)$-pure for all $\mu \leq \aleph$.

(4) For any non-Noetherian R, choose any regular cardinal $\aleph_0 \leq \aleph$ such that R does not satisfy the $\aleph^<$-A.C.C., and that $L_0 \subset L_1 \subset \cdots \subset L_\alpha \subset \cdots \subseteq L = \cup_{\alpha<\aleph} L_\alpha \leq R$ is a smooth properly ascending chain of right ideals. Then $\prod_{\alpha<\aleph}^{<\aleph} E(L/L_\alpha)$ is absolutely $(\mu^<, \aleph^<)$-pure for all $\mu \leq \aleph$, but not injective.

(5) For any cardinal $\kappa \geq \operatorname{cof} \aleph$, and any absolutely $(\mu^<, \aleph^<)$-pure A_γ, $\gamma \in \Gamma$, $\prod_{\gamma\in\Gamma}^{\mathscr{F}} A_\gamma$ is absolutely $(\mu^<, \aleph^<)$-pure.

Proof. (1) Let $\mathscr{S} : \sum_{i\in I} X^i r_{ij} = d^j \in A$, $j \in J$ be any $(\mu^<, \aleph^<)$-system over A which has a solution $X^i = (X^i_\gamma)_{\gamma\in\Gamma} = b^i = (b^i_\gamma)_{\gamma\in\Gamma} \in M$, $i \in I$. For any fixed $\gamma \in \Gamma$, $\sum X^i_\gamma r_{i1} = d^j_\gamma \in A_\gamma < M_\gamma$ has a solution $X^i_\gamma = a^i_\gamma \in A_\gamma$, $i \in I$. For each fixed $i \in I$, now define $a^i = (a^i_\gamma)_{\gamma\in\Gamma} \in M$, where $a^i_\gamma = 0$ for $\gamma \in \Gamma\backslash\Delta$ where $\Delta = \cup_{j\in J} \operatorname{supp} d^j$. Since $|J| < \aleph \leq \operatorname{cpl}(\mathscr{F})$, $\Gamma\backslash \operatorname{supp}(a^i)_{\gamma\in\Gamma} \supseteq \Gamma\backslash\Delta = \cap_{j\in J}(\Gamma\backslash \operatorname{supp} d^j) \in \mathscr{F}$. Hence $a^i \in \prod_{\gamma\in\Gamma}^{\mathscr{F}} A_\gamma$, $i \in I$, is a solution of $\mathscr{S}$ in A.

(2) If $M_\gamma = \widehat{A}_\gamma$, then $A \leq \widehat{A} \leq M = \widehat{M}$. By (1) above, $A < M$ and hence also $A < \widehat{A}$ is $(\mu^<, \aleph^<)$-pure, and thus by 4.2(1), A is absolutely $(\mu^<, \aleph^<)$-pure.

(3) Any injective module is $(\aleph^<, \aleph^<)$-pure (by 4.2(1)), and thus now A is absolutely $(\mu^<, \aleph^<)$-pure by (2) above for $\mu \leq \aleph$.

(4) By the proof of Theorem II in [D2, pp. 187–188, 4.3], $\prod_{\alpha<\aleph}^{<\aleph} E(L/L_\alpha)$ is not injective, but by (3) above, it is absolutely $(\aleph^<, \aleph^<)$-pure and hence absolutely $(\mu^<, \aleph^<)$-pure for $\mu \leq \aleph$.

(5) Let $\mathscr{F} = \{\Delta \mid \Delta \subseteq \Gamma, |\Gamma\backslash\Delta| < \kappa\}$. For any subset $J \subseteq \mathscr{P}(\Gamma)$ with $|J| < \aleph$, $|\cap J| \leq \sum\{|\Gamma\backslash\Delta| \mid \Delta \in J\} < \kappa$ because $|J| < \aleph \leq \operatorname{cof} \kappa$. Thus $\mathscr{F}$ is $\aleph^<$-complete, and $\operatorname{cpl}(\mathscr{F}) \geq \aleph$. Hence $\prod_{\gamma\in\Gamma}^{\mathscr{F}} A_\gamma = \prod_{\gamma\in\Gamma}^{<\kappa} A_\gamma$ is absolutely $(\mu^<, \aleph^<)$-pure by (2). ∎

In view of [D2, p. 187, Theorem 4.1], we expect the last construction 3.8 to be most useful when $\sigma(R) \geq \aleph^+$ (2.7). The next few results concentrate on rings R of the general type $\sigma(R) \leq \aleph^+$. The next theorem generalizes the known result that R is Noetherian if and only if absolutely pure modules are injective ([ES, p. 268, Prop. 3.24, Me, p. 564, Theorem 3]; for R a commutative Dedekind domain, see [Ma, p. 156, Theorem 1]).

3.9. THEOREM. *For any fixed regular cardinal $\aleph \geq \aleph_0$, the following five conditions are all equivalent.*

(1) *R satisfies the $\aleph^<$-A.C.C.*

(2) $\forall\, A$, *A is absolutely* $(2^<, \aleph^<)$*-pure* $\Longrightarrow$ *A is injective.*

(3) $\exists\, \mu, 2 \le \mu \le \aleph$ *such that* $\forall\, A$, *A is absolutely* $(\mu^<, \aleph^<)$*-pure* $\Longrightarrow$ *A is injective.*

(4) $\forall\, A$, *A is absolutely* $(\aleph^<, \aleph^<)$*-pure* $\Longrightarrow$ *A is injective.*

(5) $\forall\, A$, *if for some* $2 \le \nu = \nu(A) \le \aleph$, *$A$ is absolutely* $(\nu^<, \aleph^<)$*-pure, then* $\Longrightarrow$ *A is injective.*

3.10. COROLLARY. *Above, the implications* (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) *and* (1) $\Longrightarrow$ (2) $\Longrightarrow$ (5) *hold for any* $\aleph \ge \aleph_0$, *singular or regular.*

Proof. First, $(\aleph^<, \aleph^<)$-purity implies $(\mu^<, \aleph^<)$-purity for all $\mu \le \aleph$. Hence (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) as well as (2) $\Longrightarrow$ (5).

(1) $\Longrightarrow$ (2): Given $L < R$, and $\phi : L \longrightarrow A$ as in Baer's criterion. Write $L = \sum\{r_i R \mid j \in J\}$, $|J| < \aleph$. Then the system $xr_j = \phi r_j$, $j \in J$ is consistent if and only if function ϕ is an R-homomorphism. Since as given in (2), A is absolutely $(2^<, \aleph^<)$-pure, this consistent system has a solution $y \in A$ by 3.5 (5). Thus $A = \widehat{A}$.

(4) $\Longrightarrow$ (1) and (5) $\Longrightarrow$ (1): If not, there exists an $(\aleph^<, \aleph^<)$-absolutely pure $A = \prod_{\alpha<\aleph}^{<\aleph} E(L/L_\alpha)$ exactly as in 3.5 (4) which is not injective, thus contradicting (3). Alternate proof of (4) $\Longrightarrow$ (1) and (5) $\Longrightarrow$ (1): By [D2, p. 187, 4.2], if for some cardinal $\aleph$ (regular or not), it is the case that every $\aleph^<$-product $A = \prod_{\gamma\in\Gamma}^{<\aleph} A_\gamma$ of any injective modules is injective, then R is $\aleph^<$-Noetherian. The regularity of $\aleph$ and 3.8 (3) and (4) imply that A is absolutely $(\aleph^<, \aleph^<)$-pure. Now hypothesis (4) or (5) guarantees the injectivity of all such A. ∎

Besides 3.8 (3) and 3.8 (5), another way to construct absolutely $(\mu^<, \aleph^<)$-pure submodules is to take large ascending unions. The proof is omitted, but is based on 3.5 (5).

3.11. DEFINITION (Construction). For cardinals $\aleph_0 \le \aleph$, $\mu \le \aleph$ and an ordinal $\aleph_0 \le \kappa$, let $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots < M$, $\alpha < \kappa$, be an ascending well-ordered chain of submodules of some fixed module M. Let $N = \cup\{N_\alpha \mid \alpha < \kappa\}$. If $\aleph \le \operatorname{cof} \kappa$, then the following hold.

(i) If all $N_\alpha < M$ are $(\mu^<, \aleph^<)$-pure, then so is also $N < M$.

(ii) If all N_α are absolutely $(\mu^<, \aleph^<)$-pure, then so is also N.

The finite $(\aleph_0^<, \aleph_0^<)$-case of the next theorem is in [E, p. 362].

3.12. THEOREM. *For a finite or infinite cardinal* $\aleph$, *let M be a given R-module with* $\operatorname{gen} M < \aleph$, *and assume that for every absolutely* $(\aleph^<, \aleph^<)$*-pure module A,* $\operatorname{Ext}_R^1(M, A) = 0$. *Then*

(i) $\operatorname{cof}(\operatorname{rel} M) < \aleph$; *in particular if*

(ii) $\operatorname{rel} M$ *is regular, then* M *is* $(\mu^<, \aleph^<)$*-presented, where* $\mu = (\operatorname{gen} M)^+$.

Proof. Let $0 \longrightarrow G \longrightarrow F \longrightarrow M \longrightarrow 0$ be exact, where $F = \oplus\{e_i R \mid i \in I\}$ is free with $|I| = \operatorname{gen} F = \operatorname{gen} M < \mu$ on $\{e_i \mid i \in I\}$. We may view a minimal cardinality generating set $X \subset F$ of G as a well-ordered cardinal number $X = |X|$. Take $\aleph_0 \leq |X|$, otherwise we are done. For $x, y, z \in X$, define $G_x = \sum\{zR \mid z \in X, z < x\}$, and the ascending union $A = \cup_{x\in X} \prod_{y\leq x} E(G/G_y) < \prod_{x\in X} E(G/G_x)$ of direct summands of the latter, as in 3.11 (ii). If $g \in G_x$, then for any $x < z \in X$, $g \in G_x \subset G_z$, and hence $g + G_z = 0$ for all $z \geq x$. Hence $(g + G_y)_{y\in X} \in \prod_{y\leq x} E(G/G_y) < A$. Thus there is a monomorphism $\alpha : G \longrightarrow A$, $\alpha g = (g + G_x)_{x\in X}$. Since $\operatorname{rel} M = |X| = \kappa$, if $\operatorname{cof} \kappa < \aleph$, (i) and (ii) follow. So assume $\aleph \leq \operatorname{cof} \kappa \leq \kappa = |X|$. By 3.11 (ii) with $\kappa = |X| > \operatorname{cof} \kappa \geq \aleph$, A is absolutely $(\aleph^<, \aleph^<)$-pure. Consequently by 3.5 (4), α extends to $\tilde{\alpha} : F \longrightarrow A$. Select any $x(i) \in X$ such that $\tilde{\alpha}(e_i) \in \prod_{y\leq x(i)} E(G/G_y)$. Since $X = |X|$ is a well-ordered cardinal, either $\sup_{i\in I} x(i) \notin X$, in which case $\sup_{i\in I} x(i) = |X|$, or $z = \sup_{i\in I} x(i) < |X|$ and $z \in X$.

(i) Since $|I| < \aleph \leq \operatorname{cof} |X|$, the last alternative holds, and there exists an x with $z < x \in X = |X|$. But $x \notin G_x$, and $x + G_x \neq 0$. Supports of all elements of $\tilde{\alpha}(F)$, such as $\tilde{\alpha}(x)$, are contained in $[0, z] \subset X$. But $x \in \operatorname{supp} x$, and $x \notin [0, z]$ then is a contradiction. Hence (i) holds. (ii) By (i), $\operatorname{cof}(\operatorname{rel} M) = \operatorname{rel} M = |X| < \aleph$. Hence $M \cong F/G$ is $(\mu^<, \aleph^<)$-present. ∎

For $\aleph = \aleph_0$, we immediately obtain [E, p. 361].

3.13. Corollary (E. Enochs). *If* M *is finitely generated and* $\operatorname{Ext}^1_R(M, A) = 0$ *for all absolutely pure modules* A, *then* M *is finitely presented.*

3.14. Corollary. *For any cardinal* $\aleph \leq \aleph_\omega$, *and a given* R*-module* M *with* $\operatorname{gen} M < \aleph$, *assume also in addition that* $|R| < \aleph_\omega$. *Then* M *is* $(\aleph^<, \aleph^<)$*-presented if and only for every absolutely* $(\aleph^<, \aleph^<)$*-pure module* A, $\operatorname{Ext}^1_R(M, A) = 0$.

Proof. $\Longleftarrow$: by 3.5 (3). $\Longrightarrow$: In general, for any module M over any ring R, always $\operatorname{rel} M \leq (\operatorname{gen} M) \cdot |R| \cdot \aleph_0$. Consequently, now $\operatorname{rel}(M) < \aleph_\omega$, and by 3.12(i), $\operatorname{rel} M = \operatorname{cof}(\operatorname{rel} M) < \aleph$. ∎

3.15. Theorem. *For cardinals* $2 \leq \mu, \aleph$, $\mu \leq \aleph_0 \cdot \aleph$, *and where* $\mu < \aleph_0$ *if* $\aleph < \aleph_0$, *consider the following three properties of the ring* R.

(1) *Any reduced product of injective modules modulo a filter* $\mathcal{F}$ *with* $\operatorname{cpl}(\mathcal{F}) \geq \aleph$ *is absolutely* $(\mu^<, \aleph^<)$*-pure.*

(2) $\forall$ *set* I, $|I| < \mu$, $\forall\, G < R^{(I)}$, $\operatorname{gen} G < \aleph \Longrightarrow \operatorname{rel} G < \aleph$.

(3) $\forall\, I$, $|I| < \mu$, $\forall\, G < R^{(I)}$, $\operatorname{gen} G < \aleph \Longrightarrow \operatorname{cof}(\operatorname{rel} G) < \aleph$.

Then (2) $\Longrightarrow$ (1) $\Longrightarrow$ (3).

Proof. First, some common notation is established for both parts of the proof. There is a presentation

$$G = \left\langle y_j, j \in J \mid \sum_{j\in J} y_j s_{jk} = 0, k \in K \right\rangle < R = \bigoplus_{i\in I} e_i R$$

with $\operatorname{gen} G = |J|$ and $\operatorname{rel} G = |K| = K = [0, K)$, where the latter index set is viewed as a cardinal number. Also $R^{(I)}$ is free on $\{e_i\}$ and let $R^{(J)} = \oplus_{j\in J} \varepsilon_j R$ be free on $\{\varepsilon_j\}$. Define $h_k = \sum_{j\in J} \varepsilon_j s_{jk}$, and $H = \langle \{h_k \mid k \in K\}\rangle < R^{(J)}$. Then $G \cong R^{(J)}/H$ is a presentation of G, where $y_j \longrightarrow \varepsilon_j + H$ induces the above isomorphism. We may assume that for any $0 < k \leq k(2) \in K$, $h_{k(2)} \notin \langle h_\gamma \mid \gamma < k\rangle < R^{(J)}$.

(2) $\Longrightarrow$ (1). Let $Q = \prod_{\gamma\in\Gamma} F_\gamma/\mathscr{F}$, where Γ is any set, the F_γ are injective, and $\operatorname{cpl}(\mathscr{F}) \geq \aleph$. Let $\rho : P = \prod_{\gamma\in\Gamma} F_\gamma \longrightarrow Q$ and $\pi_\gamma : P \longrightarrow F_\gamma$ be the canonical quotient maps. By 3.5 (4), it suffices to show that any homomorphism $\alpha : G \longrightarrow Q$ extends to $\tilde{\alpha} : R^{(I)} \longrightarrow Q$ with $\tilde{\alpha} \mid G = \alpha$. Choose elements $fy_j \in P$ with $\rho f y_j = \alpha y_j$. Then since unlike f, ρ and α are R-homomorphisms,

$$\rho\left[\sum_{j\in J}(fy_j)s_{jk}\right] = \sum_{j\in J}(\rho f y_j)s_{jk} = \alpha\left[\sum_{j\in J} y_i s_{jk}\right] = \alpha(0) = 0.$$

Consequently, for each $k \in K$, there is a $\Gamma_k \in \mathscr{F}$ (depending on f) such that

$$\pi_\gamma\left[\sum_{j\in J}(fy_i)s_{jk}\right] = 0 \quad \text{for all} \quad \gamma \in \Gamma_k.$$

By property (2), $\operatorname{rel} G = |K| < \aleph \leq \operatorname{cpl}(\mathscr{F})$, and hence $\Gamma(*) = \cap\{\Gamma_k \mid k \in K\} \in \mathscr{F}$. Let $\pi_{\Gamma(*)} : P \longrightarrow \prod\{F_\gamma \mid \gamma \in \Gamma(*)\} \hookrightarrow P$ be the canonical projection onto $\Gamma(*)$ followed by the natural inclusion into P. Thus $\rho f = \rho\pi_{\Gamma(*)} f$. Then $\pi_{\Gamma(*)} f$ is defined on all the y_i and preserves all the relations of the y_js. Hence $\pi_{\Gamma(*)} f$ extends by R-linearity to an R-module homomorphism also denoted as before by $\pi_{\Gamma(*)} f : G \longrightarrow P$. Since P is injective, $\pi_{\Gamma(*)} f$ extends to an R-map $g : R^{(I)} \longrightarrow P$ with $g \mid G = \pi_{\Gamma(*)} f$. Then define $\tilde{\alpha}$ by $\tilde{\alpha} = \rho g$. Since $\tilde{\alpha} y_j = \rho(g y_j) = \rho(\pi_{\Gamma(*)} f y_j) = \rho f y_j = \alpha y_i$ for all $j \in J$, $\tilde{\alpha}|G = \alpha$. By 3.5(4), Q is absolutely $(\mu^<, \aleph^<)$-pure.

(1) $\Longrightarrow$ (3): If not, then in the notation of the beginning of this proof, there is a $G < R^{(I)}$ with $\operatorname{gen} G = |J| < \aleph$, but $\aleph \leq \operatorname{cof}(\operatorname{rel} G) \leq \operatorname{rel} G = |K| = K$. For $k \in K$, form $F_k = E(R^{(J)}/\langle\{h_\gamma \mid \gamma < k\}\rangle)$. Define $P = \prod\{F_k \mid k \in K\}$, $\mathscr{F} = \{\Gamma \subseteq K \mid |K\backslash\Gamma| < |K|\}$, and then form $Q = P/\mathscr{F}$. As before, retain the previous notation for $\rho : P \longrightarrow Q$ the natural quotient map, and $\pi_k : P \longrightarrow F_k$ the natural projections, $k \in K$. Let $\phi : R^{(J)} \longrightarrow P$ by $\phi\zeta = (\zeta + \langle\{h_\gamma \mid \gamma < k\}\rangle)_{k\in K} \in P$, where $\zeta = (\zeta_j)_{j\in J} \in R^{(J)}$, and

where $\pi_k \phi \zeta = \zeta + \langle\{h_\gamma \mid \gamma < k\}\rangle \in R^{(J)}/\langle\{h_\gamma \mid \gamma < k\}\rangle \in F_k$. In general, for filters of the type $\mathcal{F}$, $\operatorname{cpl}(\mathcal{F}) = \operatorname{cof}|K|$; and by the choice of G, $\aleph \leq \operatorname{cof}(\operatorname{rel} G) = \operatorname{cof}|K| = \operatorname{cpl}(\mathcal{F})$. Consequently by hypotheses (1) on our ring, Q is absolutely $(\mu^<, \aleph^<)$-pure. A contradiction of (1) will be obtained by proving that the $|I| < \mu$, $|J| < \aleph$-system of equations

$$\sum_{i\in I} X^i r_{ij} = \rho\phi\varepsilon_j = \rho\big[\big(\varepsilon_j + \langle\{h_\gamma \mid \gamma < k\}\rangle\big)_{k\in K}\big], \quad j \in J$$

is consistent, but unsolvable in Q, where $y_i = \sum_{i\in I} e_i r_{ij} \in R^{(I)}$.

Suppose that for a finite set $c_j \in R$, $j \in J(0) \subseteq J$, $|J(0)| < \infty$, $\sum_{j\in J(0)} \sum_{i\in I} X^i r_{ij} c_j = 0$. Take $c_j = 0$ for $j \in J\backslash J(0)$. Note that $\{i \in I \mid r_{ij} \neq 0 \text{ for some } j \in J(0)\}$ is a finite set, and the above is just the finite set of nonzero equations $\sum_{j\in J} r_{ij} c_j = 0$, for all $i \in I$. Then by the definition of a presentation there are at most a finite number of nonzero $t_k \in R$ for $k \in K$ such that each $c_j = \sum_{k\in K} s_{jk} t_k$, $j \in J$. Upon interchanging the order of summation, we obtain

$$\begin{aligned}\sum_{j\in J} \rho\phi\varepsilon_j c_j = \rho\phi\Bigg[\sum_{k\in K}\Bigg(\sum_{j\in J}\varepsilon_j s_{jk}\Bigg)t_k\Bigg] &= \rho\phi\Bigg[\sum_{k\in K} h_k t_k\Bigg] \\ &= \rho\phi\big[h_{k(1)}t_{k(1)} + \cdots + h_{k(m)}t_{k(m)}\big],\end{aligned}$$

where $k(1) < \cdots < k(m) \in K$. For $k(m) < k \in K$, and for any $\lambda \leq k(m)$, $\pi_k \phi h_\lambda = h_\lambda + \langle\{h_\gamma \mid \gamma < k\}\rangle = 0$ because $h_\lambda \in \langle\{h_\gamma \mid \gamma < k\}\rangle$ for all $k \in [0, K)\backslash[0, k(m)] \in \mathcal{F}$. Consequently, also $\sum_{j\in J} \rho\phi\varepsilon_j c_j = 0$, and our system of equations is consistent.

Assume that for some $\xi^{(i)} = (\xi_k^{(i)})_{k\in K} \in P$, $X^i = \rho\xi^{(i)} \in Q$, $i \in I$, is a solution with $\sum_{i\in I} \rho\xi^{(i)} r_{ij} = \rho\phi\varepsilon_j$, $j \in J$. This means that for each $j \in J$, there is an element $\Gamma_j \in \mathcal{F}$ such that for all $\lambda \in \Gamma_j$, $\pi_\lambda \sum_{i\in I} \xi^{(i)} r_{ij} = \pi_\lambda \phi\varepsilon_j$. In view of $\aleph \leq \operatorname{cpl}(\mathcal{F})$, there exists a $\lambda \in \cap\{\Gamma_j \mid j \in J\} \in \mathcal{F}$. Thus for this one λ, now $\pi_\lambda \sum_{i\in I} \xi^{(i)} r_{ij} = \pi_\lambda \phi\varepsilon_j$ holds for all $j \in J$. We now conclude the proof by finding an R-linear combination which makes the left side zero but the right side not zero in the last equation. The relation $0 = \sum_{j\in J} y_j s_{jk} \in R^{(I)}$, for $k \in K$, entails that each component separately is $0 = \sum_{j\in J} r_{ij} s_{jk} \in R$ for all i and all $k \in K$. Thus for any completely arbitrary $k \in K$,

$$\pi_\lambda \sum_{i\in I} \xi^{(i)} \cdot 0 = \pi_\lambda \sum_{i\in I} \xi^{(i)}\Bigg(\sum_{j\in J} r_{ij} s_{jk}\Bigg) = \sum_{j\in J}\Bigg(\sum_{i\in I} \pi_\lambda \xi^{(i)} r_{ij}\Bigg) s_{jk}.$$

For the above to hold, $\lambda \in K$ could be arbitrary, but from now on $\lambda \in \cap_{j\in J}\Gamma_j$. This allows us to replace the last sum on I by $\pi_\lambda \sum_{i\in I} \xi^{(i)} r_{ij} = \pi_\lambda \phi\varepsilon_j$ as follows

$$0 = \sum_{j\in J} \pi_\lambda \phi\varepsilon_j s_{jk} = \pi_\lambda \phi\Bigg(\sum_{j\in J} \varepsilon_j s_{jk}\Bigg) = \pi_\lambda \phi h_k, \ k \in K.$$

But if $\lambda \leq k$, then $h_k \notin \langle \{h_\gamma \mid \gamma < \lambda\} \rangle$, and $0 \neq \pi_\lambda \phi h_k = h_k + \langle h_\gamma \mid \gamma < \lambda \rangle \in R^{(J)}/\langle \{g_\gamma \mid \gamma < \lambda\} \rangle$ is a contradiction. Hence $\operatorname{cof}(\operatorname{rel} G) < \aleph$. ∎

For $\mu = 2$ and $\aleph = \aleph_0$, our last condition 3.15(1) implies that the ring R must be coherent below.

3.16. COROLLARY [ES]. *Assume that* 3.15(1) *holds for* $\mu = 2$ *and some* $2 \leq \aleph \leq \aleph_0$. *Then for any* $L < R$ *and for any* $n < \aleph$, *if* $\operatorname{gen} L < n$, *then also* $\operatorname{rel} L < \aleph_0$ *is finite. In particular, if* $\aleph = \aleph_0$, *then every finitely generated right ideal of* R *is finitely presented.*

Proof. Absolutely $(2^<, \aleph^<)$-pure is a synonym of "$\aleph$-injective" as used in [ES]. By 3.15(1), condition (ii) in [ES, p. 261, Theorem 3.12] holds. Now use the condition (iii) of the latter together with [ES, p. 263, Lemma 3.13]. ∎

3.17. COROLLARY. *For any cardinal* $\aleph \leq \aleph_\omega$, *assume that* $|R| < \aleph_\omega$. *Then in Theorem* 3.15, (1) $\iff$ (2).

Proof. It suffices to show that (3) $\Longrightarrow$ (2) in 3.15. First, if $\aleph < \aleph_0$, (3) $\iff$ (2). So let $\aleph_0 \leq \aleph$, and hence $\mu \leq \aleph \cdot \aleph_0 = \aleph \leq \aleph_\omega$, and $|I| < \mu \leq \aleph_\omega$. Let $\operatorname{Fin} \mathscr{P}(I) = \{A \subseteq I \mid |A| < \aleph_0\}$. Then

$$|G| \leq \left|R^{(I)}\right| \leq |\operatorname{Fin} \mathscr{P}(I)| \cdot |R| \leq |I| \cdot \aleph_0 \cdot |R| < \aleph_\omega.$$

For any module G,

$$\operatorname{rel} G \leq |\operatorname{Fin} \mathscr{P}(G)| \cdot |R| \leq |G| \cdot \aleph_0 \cdot |R|.$$

Thus now $\operatorname{rel} G < \aleph_\omega$. Hence $\operatorname{cof}(\operatorname{rel} G) = \operatorname{rel} G$, and in 3.15, (3) $\iff$ (2). ∎

4. $\aleph^<$-NOETHERIAN RINGS

Two new characterizations of $\aleph^<$-Noetherian rings are obtained, in addition to Theorem 3.9. First, for these $\aleph^<$-Noetherian rings, some conditions on the generators of free modules are obtained.

4.1. PROPOSITION. *Suppose that* R *satisfies the* $\aleph^<$-*A.C.C. for some* $\aleph_0 \leq \aleph$, *and that* $G \leq F$ *is a submodule of a free module* F. *Then*

(i) $\operatorname{gen} G \leq \aleph \cdot \operatorname{gen} F$; *moreover*

(ii) *if* $\operatorname{gen} F < \aleph$, *and* $\aleph$ *is regular, then* $\operatorname{gen} G < \aleph$.

Proof. (i) and (ii): Let $F = \oplus_{i\in I} e_i R$ be free on $\{e_i\}_{i\in I}$, where I is a cardinal number $I = |I| \geq \aleph_0$. For any $i \in I$, define $F_{<i} = \oplus\{e_k R \mid k < i\} < F_i = \oplus\{e_k R \mid k \leq i\}$ and similarly $G_{<i} = G \cap F_{<i} \leq G_i = G \cap F_i$. View $G_i/G_{<i} \cong (G_i + F_{<i})/F_{<i} = L(i) \leq R$. Then $\operatorname{gen} L(i) = \tau(i) < \aleph$ by the $\aleph^<$-A.C.C. For $i = 0$, $G_{<0} = (0) < G_0 = e_0 R \cap G = e_0 L(0) = \langle\{g_{0j} \mid j < \tau(0)\}\rangle$. By ordinal induction assume that for some ordinal $i \in I = |I|$, $g_{\mu j} \in G_\mu$ have already been selected (for $\mu < i, j < \tau(\mu)$) so that for any $k < i$, $G_k = \langle\{g_{\mu j} \mid j < \tau(\mu), \mu \leq k\}\rangle$, where $\langle\{(g_{\mu j} + F_{<\mu})/F_{<\mu} \mid j < \tau(\mu)\}\rangle = L(\mu)$ is a set of generators of $L(\mu)$, $\mu \leq k < i$. Always, $G_{<i} = \cup\{G_\mu \mid \mu < i\}$. Now select any generators $g_{ij} \in G_i$ for $L(i)$ so that $L(i) = (G_i + F_{<i})/F_{<i} = \langle\{g_{ij} + F_{<i} \mid j < \tau(i)\}\rangle$. For any $\xi \in G_i$, for some finite number of $t_j \in R$, $\xi - \sum_j g_{ij} t_j \equiv v \in F_{<i} \cap G = \cup\{G_i \cap G \mid k < i\}$. By induction, v is a linear combination of g_{kj}, $k < i$. Hence $G_i = \langle\{g_{kj} \mid j < \tau(k), k \leq i\}\rangle$. Lastly, $G = \cup_{i\in I} G_i = \langle\{g_{ij} \mid j < \tau(i), i \in I\}\rangle$. Consequently, $\operatorname{gen} G \leq \sum\{\tau(i) \mid i \in I\} = |I| \cdot \sup_{i\in I} \tau(i)$. Since all $\tau(i) < \aleph$, and $|I| = \operatorname{gen} F$, (i) $\operatorname{gen} G \leq \aleph \cdot \operatorname{gen} F$ follows, and if $\aleph$ is regular, (ii) $\operatorname{gen} G < \aleph$.

In the finite case $\aleph = \aleph_0$, the next theorem is already known, see [Wi, p. 223, 27.3].

4.2. THEOREM. *For an infinite regular cardinal $\aleph$ and any ring R, the following are equivalent*:

(i) *R satisfies the $\aleph^<$-A.C.C.*

(ii) *For any module M, $\operatorname{gen} M < \aleph \Longrightarrow \operatorname{rel} M < \aleph$.*

Proof. First, for any M take a free module F and $G < F$ with $M \cong F/G$, with $\operatorname{rel} M = \operatorname{gen} G$, and with either both $\operatorname{gen} M$ and $\operatorname{gen} F = \operatorname{rank} F$ finite, or with $\aleph_0 \leq \operatorname{gen} M = \operatorname{gen} F$. (i) $\Longrightarrow$ (ii). If $\operatorname{gen} M < \aleph$, than also $\operatorname{gen} F < \aleph$, since $\aleph_0 \leq \aleph$. Now by 4.1 (ii), $\operatorname{rel} M = \operatorname{gen} G < \aleph$.

(ii) $\Longrightarrow$ (i). By [D3, p. 2882, Lemma 2.4], if $\aleph$ is regular, R satisfies the $\aleph^<$-A.C.C. if and only if for any $L \leq R$, $\operatorname{gen} L < \aleph$. Thus if R is not $\aleph^<$-Noetherian, then there exists a $G < R$ with $\operatorname{gen} G = \aleph = \operatorname{rel}(R/G)$. The latter holds because any minimal set of relations for the generator $1 + G$ of R/G is of the form $(1 + G)g_j = G$, $j \in J$, where $G = \sum\{g_j R \mid j \in J\} = \langle\{g_j \mid j \in J\}\rangle$. But then $1 = \operatorname{gen} R/G < \aleph$ with $\operatorname{rel} R/G = \aleph$ is a contradiction of (ii). Hence R is $\aleph^<$-Noetherian. ∎

4.3. COROLLARY. *For any ring R, let as before $\sigma(R)$ be the unique smallest infinite cardinal such that R satisfies the $\sigma(R)^<$-A.C.C. Then*

(i) *for any regular $\kappa < \sigma(R)$ cardinal κ, the free module $R^{(\kappa)}$ of rank κ contains a submodule $G < R^{(\kappa)}$ with* $\operatorname{gen} G \geq \kappa$.

(ii) *Hence if $\sigma(R)$ is a limit cardinal, then* $\sigma(R) = \sup\{\kappa \mid \kappa \text{ is regular}; \exists\, G < R^{(\kappa)}, \operatorname{gen} G \geq \kappa\}$.

4.4. Corollary. *For any regular cardinal $\aleph \geq \aleph_0$, (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) below.*

(i) *R satisfies the $\aleph^{<}$-A.C.C.*

(ii) *Any reduced product of injective modules modulo a filter $\mathcal{F}$ with $\operatorname{cpl}(\mathcal{F}) \geq \aleph$ is injective.*

(iii) *Any reduced product of injective modules modulo a filter $\mathcal{F}$ with $\operatorname{cpl}(\mathcal{F}) \geq \aleph$ is absolutely $(\aleph^{<}, \aleph^{<})$-pure.*

Proof. (i) $\Longrightarrow$ (ii). P. Loustaunau has shown that 4.4 (1) is equivalent to the condition that every $\mathcal{F}$-product of injective modules is injective, for $\mathcal{F}$ with $\operatorname{cpl}(\mathcal{F}) \geq \aleph$ ([Lo4, p. 3676, Theorem 2.3]). If for $\aleph \geq \aleph_0$ regular, R satisfies the $\aleph^{<}$-A.C.C., then in the short exact sequence in 1.4, the first two terms are injective. Hence their quotient, the reduced product in (ii), is also injective. Trivially, (ii) $\Longrightarrow$ (iii). ∎

If for $\aleph_0 \leq \aleph$ regular, R satisfies the $\aleph^{<}$-A.C.C., and $F_\gamma, \gamma \in \Gamma$ are injective modules, it would be interesting to see in the light of [Lo 4] just how $\prod_\Gamma F_\gamma/\mathcal{F}$ is a direct summand of $\prod_\Gamma F_\gamma$.

4.5. *Conjecture.* If $\aleph$ is measurable, then in 3.15 and 4.4 more equivalent statements can be added where "reduced product" is replaced by "ultra-product" (see [Lo 4, p. 3677, Note 2]).

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